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A system level model reduction approach for flexible multibody systems with parametric uncertainties.

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Abstract

Stochastic analysis of flexible multibody system for uncertain parameters typically requires a large number of simulation runs for example for Monte-Carlo simulation. However, as the computational load of a regular flexible multibody model is typically rather high, this is often infeasible. A solution to this high computational load is model reduction, but regular model reduction approaches for flexible multibody simulation do not maintain the parameter dependency. This leads to a new model reduction for each parameter which also leads to high computational costs. The current work presents a novel system level model reduction technique for parameterized flexible multibody simulation. The proposed approach is a parameterized version of the Global Modal Parameterization method. In this approach a system level model reduction of the flexible mechanism is performed in which a configuration dependent projection space is used. For the parameterized approach, affine parameter dependence is assumed. In this case the parameter dependency can be externalized and is exactly preserved through the model reduction. The accuracy of the proposed approach is demonstrated through a numerical validation. The model is used for a Monte-Carlo simulation of mechanism with uncertain parameters and delivers accurate probabilistic distributions for the motion of the mechanisms at a highly reduced cost compared to the original model. The proposed approach is shown to provide reliable results with a computational load which is reduced from days to hours.

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Keywords: Flexible multibody simulation; model reduction; parametric; Global Modal Parameterization

1. Introduction

Over the years, flexible multibody simulation has become an important tool to evaluate mechanical machine design. However, in practice machines always have uncertainties, e.g. due to design tolerances. It is very difficult to assess the effects of these uncertainties during the design. In the past many approaches have been developed to analyze uncertainties in linear structures, but the analysis of flexible multibody systems is considerably more complicated due to the nonlinear behavior. This leads to the need of Monte-Carlo type methods, in which a large number of simulations

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has to be run for a range of parameters. Coupled to the high computational load of flexible multibody models, this leads to infeasible computational demands.

In this research, model reduction is proposed for reducing the computational load to an acceptable level for uncertainty evaluations. Over the past decade(s) considerable research effort has been spent on the development of component-level model reduction techniques, such as floating-frame-of-reference component mode synthesis (FFR-CMS)[1]. More recently a system level model reduction technique, Global Modal Parameterization, was developed which greatly increases computational efficiency over component level methods[2,3]. However these existing model reduction approaches do not support parameter dependence, which means a new reduced model should be constructed for each parameter, which would prove to be numerically very costly. Several researchers have been focusing on the development of parametric model order reduction techniques[4]. For general parameterizations, the sampling-interpolation based methods seem the most promising. In this framework different sampling and interpolation schemes have been developed which enable the efficient evaluation of parameterized linear reduced models.

In this research the two previously mentioned reduced modeling approaches are merged in order to create a parametric nonlinear system level model reduction technique for flexible multibody simulation. This new method is referred to as *parametric Global Modal Parameterization* (pGMP). This reduction approach consists of several steps:

- *Pre-processing*: During this phase, a sample space for the configurations of the mechanism is first constructed, as described by the GMP formalism. For this grid the linearized system matrices are computed and reduced according to linear model reduction schemes. For an affine parameter dependence, the partial matrices are reduced separately such that the reduced model maintains the parameterization.
- *Simulation*: In the case of uncertainty evaluation of nonlinear systems, the simulation is often based on a Monte-Carlo approach. However the visited points might not be present in the sampling grid, such that interpolation between the locally reduced models is required. During the simulation run, a dynamic interpolation is applied between the locally reduced system matrices, which allows to recover the nonlinear behavior of the multibody system. For the parameter dependency, no interpolation has to be performed because the affine parametric functions can be evaluated with the reduced matrices.
- *Post-processing*: Finally the reduced models can be projected back onto their unreduced degrees-of-freedom and the results of interest can be analyzed.

In the first section the original model definition is discussed and a set of models which can be treated is presented. Secondly the novel model reduction scheme is presented, with special attention for the parametric reduction.

The proposed approach is demonstrated on a numerical example of a flexible planar slider-crank mechanism. The system has uncertain crank length and is subject to a known input torque on the crank. A probability distribution of the output position of the slider is constructed through a basic Monte-Carlo simulation using both the unreduced and reduced model. The proposed pGMP approach is shown to provide a considerable speed up even for this relatively simple example while still providing good accuracy.

2. Original model definition

In this work a model reduction approach for flexible multibody systems is introduced. In order for this reduction approach to be applicable, the flexible multibody model must have a predefined structure. In general the equations of a parameterized multibody system can be summarized as:

$$
\begin{aligned}
&M(p,q)\ddot{q} + g_{gyr}(p,q,\dot{q}) + g_{int}(p,q) - \left(\frac{\partial c(q)}{\partial q}\right)^T \lambda = g_{ext},\\
&c(q) = 0.
\end{aligned}
\tag{1}
$$

In this set of equations q is a vector of multibody coordinates, p is a vector of parameters and λ is a vector of Lagrange multipliers. $M(p,q)$ is the mass matrix, g_{gyr}, g_{int} and g_{ext} are respectively the gyroscopic, internal and external force vector. Finally c is the vector of algebraic constraint equations for the connections between the different bodies. These equations are differential algebraic equations, which are costly to integrate due to the algebraic equations and the typical large number of degrees-of-freedom (DOFs). Two important assumptions which are made with respect to the parameterization are:

- no parametric dependence of the constraint equations;
- no mesh variations with the parameterization in order to have compatible DOFs for the original model.

Usually the parameter dependence is only implicitly present in multibody formulations. This is specifically true when the original multibody model is already based on a component level reduced model, like in a floating-frame-of-reference component-mode-synthesis (FFR-CMS) approach. However in this work, a description which allows for explicit parameter dependence is assumed. This kind of description can easily be obtained from any unreduced formulation like the absolute nodal coordinate formulations (ANCF)[5] or large rotation formulations[6]. Starting from an unreduced FFR approach this can also be obtained, but this is not a common description for multibody systems.

In this work, the original equations of motion can be written in the form:

$$\begin{aligned}&\sum_{i=1}^{n^M}\left(M_i(q)w_i^M(p)\right)\ddot{q}+\sum_{i=1}^{n^M}\left(g_{gyr,i}(q,\dot{q})w_i^{gyr}(p)\right)+\sum_{i=1}^{n^M}\left(g_{int,i}(q)w_i^{int}(p)\right)-\left(\frac{\partial c(q)}{\partial q}\right)^T\lambda=g_{ext},\\&c(q)=0.\end{aligned} \tag{2}$$

In this equation, the dependance on the parameters p is given through the summation of coefficient vectors and matrices multiplied by generalized weighting functions w_i^x which are a function of the parameters p, implying affine parameter dependence. Even if it is not possible to reshape the equations of motion into this shape, approximate solutions can be constructed, e.g. by using Taylor expansion around a nominal parameter value[4].

The proposed approach can be directly applied to all systems described by the above equations. However, in order to improve the readability of the following section, a more specific case is considered in this work. The demonstration case which is employed in this work is a planar large-rotation beam formulation[6,3]. The parameters which are considered, are the diameters for the beams employed. For this problem Eq. (2) becomes:

$$\begin{aligned}&\sum_{i=1}^{2}\left(M_i w_i^M(p)\right)\ddot{q}+\sum_{i=1}^{2}\left(g_{int,i}(q)w_i^{int}(p)\right)-\left(\frac{\partial c(q)}{\partial q}\right)^T\lambda=g_{ext},\\&c(q)=0.\end{aligned} \tag{3}$$

These more restrictive equations are used throughout the rest of this paper, but results can be easily generalized to the description from Eq. (2). For the case of diameter variation of the beams, only two weighting functions are necessary for both the stiffness and mass matrix and these are simple polynomial functions.

3. Parametric Global Modal Parameterization

In this work, an algorithm for the efficient evaluation of parameterized flexible multibody systems is developed. The proposed reduction method has two important properties:

- efficient evaluation of the multibody equations;
- traceability of the parameters.

In order to reach these goals, the parametric Global Modal Parameterization (pGMP) is introduced. The basis of this technique is the regular Global Modal Parameterization approach which enables efficient and even real-time flexible multibody simulation[2,7]. This is a system level nonlinear model reduction technique specifically developed for this kind of systems. Due to the system level reduction the algebraic differential equations for the original multibody model can be turned into ordinary differential equations of much smaller size. However, in the context of uncertain systems, there is not an approach to introduce parameter variations in the regular GMP approach. This would lead to the computation of a new reduced model for each parameter-set, leading to unacceptable computational costs. In order to circumvent this, an explicit parameter dependency is introduced for the pGMP formulation.

Because the GMP is a nonlinear model reduction technique it is conceptually straightforward to introduce parameter dependency as well. However, treating the parameters the same as the nonlinear DOFs leads to an exponentially rising computational load with the number of parameters and nonlinear DOFs. In order to avoid this curse of dimensionality, the parametric reduction is performed by exploiting the affine dependence discussed in the previous section. This leads to a two stage reduction for the PGMP formulation:

- Nonlinear motion reduction;
- Local parametric reduction.

By properly accounting for these two stages, an accurate and efficient formulation can be constructed for parametric flexible multibody systems. These two stages are discussed more in detail in the following sections.

3.1. Nonlinear motion reduction

The nonlinear GMP motion reduction is based on a nonlinear projection from the reduced to the unreduced DOFs q [2,3]:

$$q = \rho(\theta) + \Psi^{q\delta}(\theta)\delta. \tag{4}$$

In this function $\rho(\theta)$ is a nonlinear function which provides the undeformed configuration of the system in function of a minimal set of rigid DOFs θ. Furthermore $\Psi^{q\delta}(\theta)$ is a matrix with the configuration dependent flexible deformation modes and δ is the vector with the participation factors for these modes. The reduced coordinate vector η summarizes the GMP DOFs:

$$\eta = \begin{bmatrix} \theta \\ \delta \end{bmatrix}. \tag{5}$$

This projection has as most importing property that it is a reduction on system level which takes the (linearized) constraint equations into account [2,3], such that:

$$c\left(\rho(\theta) + \Psi^{q\delta}(\theta)\delta\right) = 0 + O(\delta^2). \tag{6}$$

In the assumption of small flexible deformation, the reduced system meets the constraint equations. Under this same assumption the velocity can be approximated as:

$$\dot{q} = \Psi^{q\eta}(\theta)\dot{\eta}. \tag{7}$$

An important assumption in the derivation of the reduced equations of motion, is the small deformation assumption. In this case the Lagrangian for the equations of motion can be approximated as:

$$\mathcal{L} = \dot{\eta}^T \Psi^{q\eta}(\theta)^T M(p) \Psi^{q\eta}(\theta)\dot{\eta} + \delta^T \Psi^{q\delta}(\theta)^T \frac{\partial g_{int}(q, p)}{\partial q} \Psi^{q\delta}(\theta)\delta - \eta^T \left(\Psi^{q\eta}(\theta)\right)^T g_{ext}. \tag{8}$$

In this equation the contribution from constraints are omitted due to Eq. (6). Based on this equation, the reduced mass and stiffness matrix are defined:

$$M^r(\theta, p) = \Psi^{q\eta}(\theta)^T M(p) \Psi^{q\eta}(\theta), \tag{9}$$

$$K^r(\theta, p) = \Psi^{q\eta}(\theta)^T \frac{\partial g_{int}(\rho(\theta), p)}{\partial q} \Psi^{q\eta}(\theta). \tag{10}$$

From this description, the energy preserving equations of motion can be derived for the reduced model. For the planar system as described above and neglecting quadratic terms in the deformation, the reduced equations of motion are:

$$M^r(\theta, p)\ddot{\eta} + g^r_{gyr}(\theta, \dot{\eta}, p) + K^r(\theta, p)\eta = \Psi^{q\eta}\theta g_{ext}, \tag{11}$$

with

$$g^{r,k}_{gyr}(\theta, \dot{\eta}, p) = \sum_{i=1}^{n^\theta} \frac{\partial M^{r,k}(\theta, p)}{\partial \theta_i} \dot{\eta}\dot{\theta}_i - \frac{1}{2}\dot{\eta}^T \frac{\partial M^r(\theta, p)}{\partial \eta_k} \dot{\eta}. \tag{12}$$

This is a set of ordinary differential equations, which can be evaluated very efficiently. Also a formulation for the configuration and parameter dependent mass and stiffness have to be defined. Because the dependence on the configuration DOF θ is in practice difficult or even impossible to obtain in closed loop form, this is taken into account by sampling the configuration space and interpolating between these samples[2,7]. However, due to the specific form of the parameter dependence discussed in the previous section, this can be accounted for in a more efficient way. This aspect is discussed in the following section.

3.2. Local parametric reduction

As discussed in Sec. 2, an affine relation exists between the unreduced mass and stiffness matrix and the parameters of the system. In the proposed approach the aim is to preserve this relationship exactly. This also circumvents the need to sample in the parametric space, which would otherwise greatly increase the computational cost of performing the model reduction.

In the proposed approach starting from Eq. (3) the reduced mass matrix becomes:

$$\begin{aligned} M^r(\theta, p) &= \Psi^{q\eta}(\theta)^T \left(\sum_{i=1}^{2} \left(M_i w_i^M(p) \right) \right) \Psi^{q\eta}(\theta) \\ &= \sum_{i=1}^{2} \left(\left(\Psi^{q\eta}(\theta)^T M_i \Psi^{q\eta}(\theta) \right) w_i^M(p) \right) \\ &= \sum_{i=1}^{2} \left(M_i^r(\theta) w_i^M(p) \right). \end{aligned} \tag{13}$$

In this case the reduced mass matrix maintains the exact parameter dependency. However, in order to take the effect of the modes changing with the configuration into account, different mass coefficient matrices M_i^r are stored for different configurations. A similar approach is adopted for the reduced stiffness matrix:

$$\begin{aligned} K^r(\theta, p) &= \Psi^{q\eta}(\theta)^T \left(\sum_{i=1}^{2} \left(\frac{\partial g_{int,i}(q)}{\partial q} w_i^{int}(p) \right) \right) \Psi^{q\eta}(\theta) \\ &= \sum_{i=1}^{2} \left(\left(\Psi^{q\eta}(\theta)^T \frac{\partial g_{int,i}(\rho(\theta))}{\partial q} \Psi^{q\eta}(\theta) \right) w_i^{int}(p) \right) \\ &= \sum_{i=1}^{2} \left(K_i^r(\theta) w_i^M(p) \right). \end{aligned} \tag{14}$$

This leads to the definition of similar stiffness coefficient matrices K_i^r, which are now also based on a configuration dependent unreduced stiffness matrix. These sets of matrices enable a very efficient evaluation of the system matrices for different parameters of the system, without a loss of accuracy in the parameter space. With these matrices, all terms necessary for the pGMP model are known. Even in the case where no explicit parameter dependence can be constructed, approximate solutions can be exploited[4,8].

3.3. Projection space

In this work a configuration dependent system level free-free modal basis which takes the constraints into account is employed. In contrast to previous work on GMP[2,9,3], the rigid body DOF θ is not fixed to compute the flexible modes. These modes are computed by employing the null space of the Jacobian of the constraint equations $N(q)$:

$$\frac{\partial c}{\partial q} N(q) = 0. \tag{15}$$

From this projection the reduction space is computed as:

$$\left(N^T K N - \Lambda N^T M N \right) \Psi^{q_{ce}\eta} = 0, \tag{16}$$

$$\Psi^{q\eta} = N \Psi^{q_{ce}\eta}. \tag{17}$$

The flexible modes are normalized with respect to the mass matrix and the rigid motion mode is normalized with respect to the rigid DOF θ. This choice of modal projection enables a full diagonalization of the reduced mass and stiffness matrix, reducing coupling effects and the computational load.

In order to get the most accurate description of the system, the reduction space $\Psi^{q\eta}$ employed should also reflect the change in dynamics space due to parameter changes. This could be obtained by making the projection space variable with respect to the parameter, but this would strongly complicate the formulation. Alternatively a constant space spanning the most important dynamics for a range or parameters could be constructed through e.g. singular value decomposition. However, this last approach would jeopardize the continuity of the reduction space in the motion space, which is of paramount importance for a correct evaluation of the dynamic equations of motion. Therefore this work employs the modal projection space based on the dynamics for a nominal parameter value, as can be often defined for uncertainty problems. Future research will focus on more robust ways to construct the reduction space for strong parameter variations.

3.4. Interpolation approach

In order to get the configuration dependent description, a function needs to be defined which describes the reduced mass and stiffness matrices as a function of the rigid parameter θ. Because it is generally impossible to find an analytical closed loop description for the reduced system matrices as a function of θ, an interpolation scheme is used to extract the properties based on a grid of precomputed points. In this work a cubic interpolation scheme is exploited. This interpolation scheme allows first order continuity over the grid points, which is required for a proper evaluation of the reduced inertial forces, as shown in Eq. (12). For the first derivatives, the central differences are used for each grid point, which leads to an Overhauser interpolation scheme.

For a given θ between θ_i and θ_{i+1}, the evaluation of a system matrix A is performed as:

$$s = (\theta - \theta_i)/(\theta_{i+1} - \theta_i), \tag{18}$$

$$A = (1 - 3s^2 + 2s^3)A_i + (3s^2 - 2s^3)A_{i+1} + (s - 2s^2 + s^3)dA_i + (-s^2 + s^3)dA_{i+1}, \tag{19}$$

with

$$dA_i = \frac{A_{i+1} - A_{i-1}}{\theta_{i+1} - \theta_{i-1}}. \tag{20}$$

This interpolation is performed for all parts of the parameter dependent matrices as defined in Eq. (13)-(14) and also for the projection matrices required for the back-transformation. These interpolated reduced matrices are then used to recover the mass and stiffness matrices for a given set of parameters. This reduction scheme thus maintains the explicit parameter dependency.

4. Numerical validation

In this section the proposed model reduction approach is validated numerically. First the accuracy of the pGMP approach is demonstrated. Then the use of the reduced model for uncertainty analysis is demonstrated in a Monte-Carlo simulation.

The system under consideration is a planar slider-crank mechanism, as shown in Fig. 1. The crank and connecting rod are circular beams of which the diameter can vary. The original model is described using a nonlinear large-rotation beam formulation, as proposed in by Géradin and Cardona [6]. The fixed properties of the model are given in Tab. 4.

The reduced model employs one rigid body degree-of-freedom θ, being the crank angle, and fifteen flexible reduction modes. The motion space is sampled with a step of $\Delta\theta = 0.01 rad$. The nominal parameters for which the reduced modes are determined are $r_1 = r_2 = 5.6mm$. In the remainder of this work only the radius of the crank r_1 is altered, and the connecting rod maintains its section $r_2 = 5.6mm$.

5. Model validation

First of all the basic dynamic behavior is compared between the unreduced and reduced model. Fig. 2 shows the first four non-zero eigenfrequencies for the unreduced and the reduced model for a full rotation of the crank for three

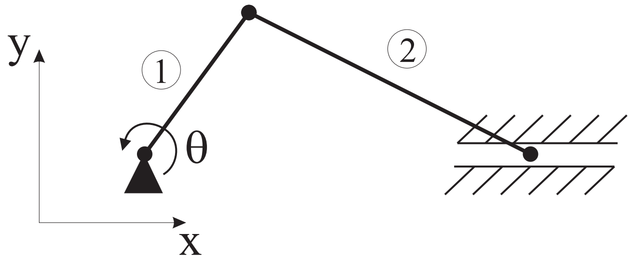

Fig. 1. Planar slider-crank mechanism for numerical validation.

Table 1. Properties of the slider crank system.

	Crank	Connecting rod
Length [m]	0.3	0.7
Young's modulus [GPa]	210	210
Density [kg/m^3]	7800	7800
# elements [/]	5	5

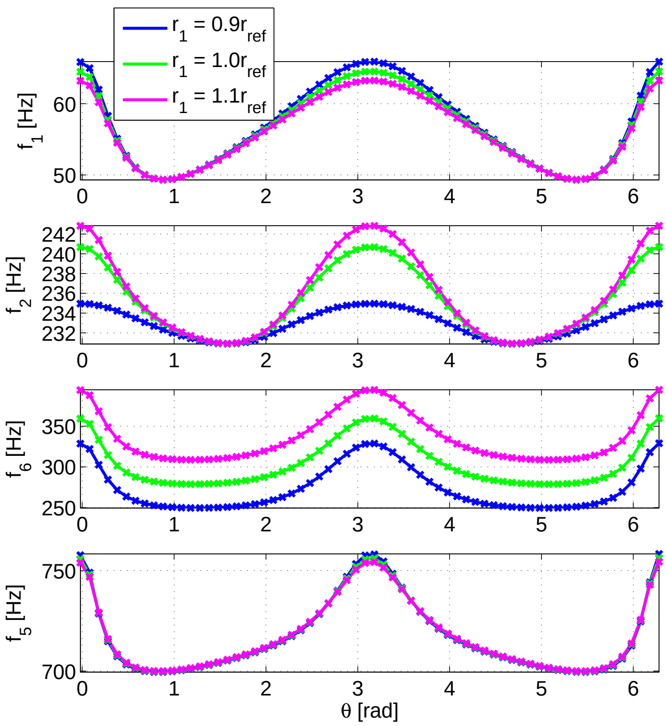

Fig. 2. Comparison of eigenfrequencies between unreduced (-) and reduced model (x).

different parameter values. This figure shows that the reduced model accurately represents the linearized dynamic behavior of the system over the full range of motion. It is important to notice that the accuracy is also very good for moderate parameter variations, even though the projection space was constructed for $r_1 = r_{ref}$.

However, for flexible multibody simulation not only the local linearized behavior has to be accurate, but also the nonlinear dynamic motion. In order to examine this case, the load torque shown in Fig. 3 is applied to the crank. For this case again three different parameter values are compared to evaluate the consistency of the model reduction for the parameter range of interest. The motion of the crank angle is shown in Fig. 4. The rigid motion is tracked very

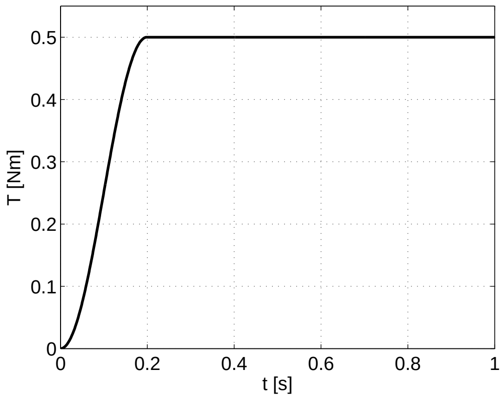

Fig. 3. Load torque on crank.

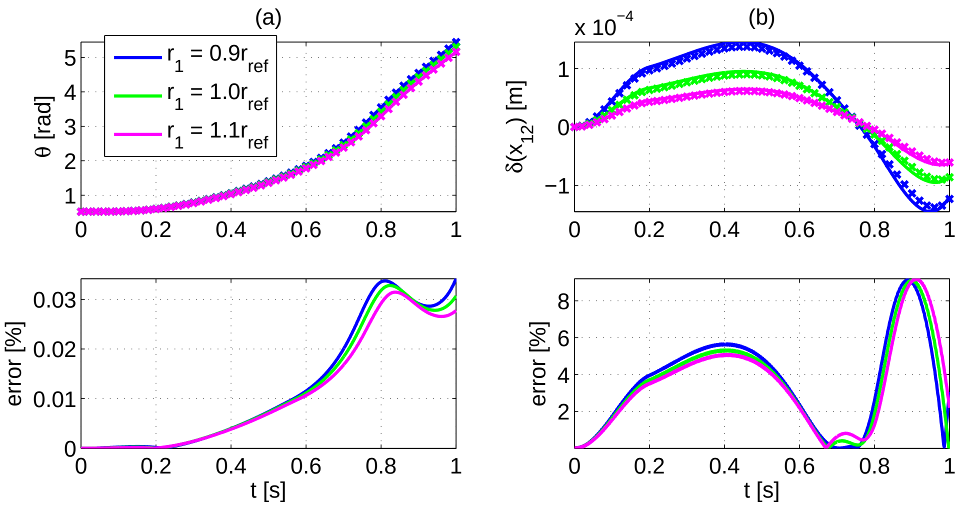

Fig. 4. Comparison of crank angle (a) and slider deformation (b) between unreduced (-) and reduced model (x).

accurately between the original and the reduced model for all three parameter values. The relative error seems to be independent of the parameter value and only determined by the magnitude of the motion, because the integrated error over a given trajectory is more important than the effect of the approximate reduction space for a given parameter.

Finally, also the flexible behavior of reduced model is validated. The flexible deformation of the slider point is shown in Fig. 4. This figure also demonstrates the good accuracy of the proposed model reduction approach for the flexible deformation of the system. The error of the flexible deformation is again mostly insensitive to the applied parameter value.

These results show that the proposed parametric GMP method is a valid approach to perform simulations of parameterized flexible multibody systems. In the following section the good accuracy and high speed of the pGMP method is exploited to perform an uncertainty analysis on the presented flexible multibody system.

6. Uncertainty analysis

To show the potential of the proposed model reduction technique, the reduced model is applied in a Monte-Carlo simulation for the slider-crank system. For this system the effect of the variation of the crank cross-section radius on the response of the system is considered. A straightforward Monte-Carlo is performed with 10000 samples created by *randn* in Matlab. The crank cross-section diameter has a normal distribution with an average of $\mu(r_1) = 5.6mm$ and a

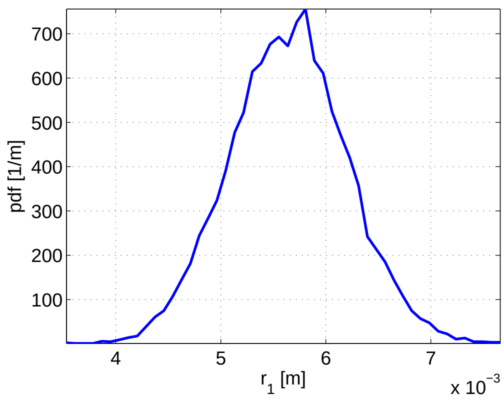

Fig. 5. Numerical probability density function for crank diameter.

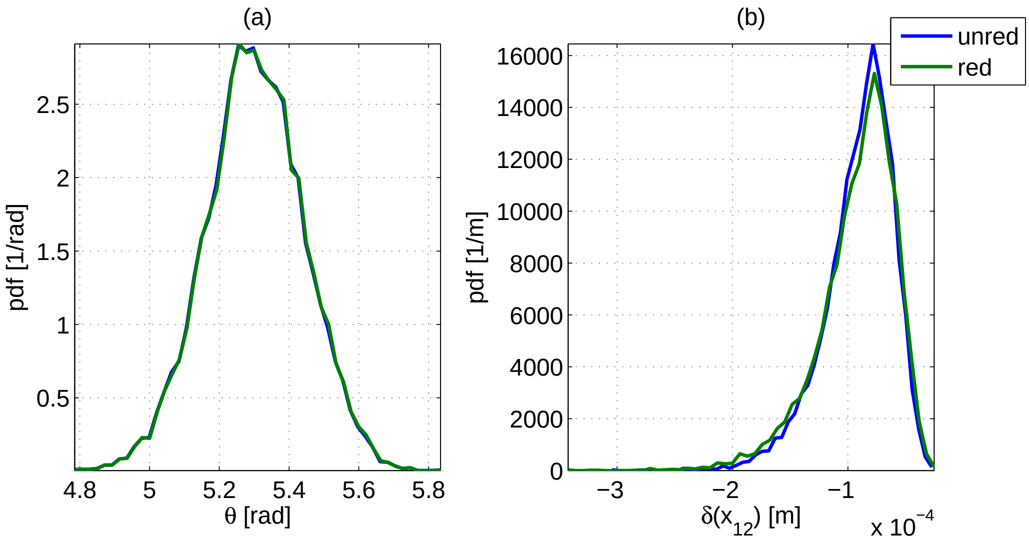

Fig. 6. Probability density function at $t = 1s$ for crank angle (a) and slider deformation (b).

ten percent standard deviation of $\sigma(r_1) = 0.56m$. With this input data the simulation is performed both for the unreduced model and the reduced model. The numerically evaluated probability density function (pdf) for the parameter r_1 (evaluated over fifty intervals) is shown in Fig. 5. The probability density function for the crank angle and the slider flexible deformation at the end of the simulation are shown in Fig. 6. These results show good correspondence for the pdf of the reduced model with the unreduced pdf, both for the crank angle and the flexible deformation of the slider.

Even for this very simple example, the computational savings of the proposed method are considerable. Both models are implemented in Matlab and the Monte-Carlo of the unreduced model takes around a week to perform, whereas the reduced model is evaluated in merely two hours. This computational advantage will be even more severe for systems of higher complexity and systems exhibiting 3D dynamics.

7. Conclusion

The current work presents a novel system level model reduction technique for parameterized flexible multibody simulation. This technique can be exploited for stochastic simulations of multibody system with uncertain parameters, in which case the original computational load is often infeasible. The proposed approach is a parametric version of the Global Modal Parameterization method. In this approach a system level model reduction of the flexible mechanism is performed in which a configuration dependent projection space is used. For the parametric approach, affine parameter dependence is assumed. In this case the parameter dependency can be externalized and is exactly preserved through the model reduction. The reduction space employed in this work is independent of the system parameters and future research should also address this. The accuracy of the proposed approach is demonstrated through a numerical validation. The model is used for a Monte-Carlo simulation of mechanism with uncertain parameters and delivers accurate probabilistic distributions for the motion of the mechanisms at a highly reduced cost compared to the original model.

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